Adventures in Tiling the Plane

Ken Cutter

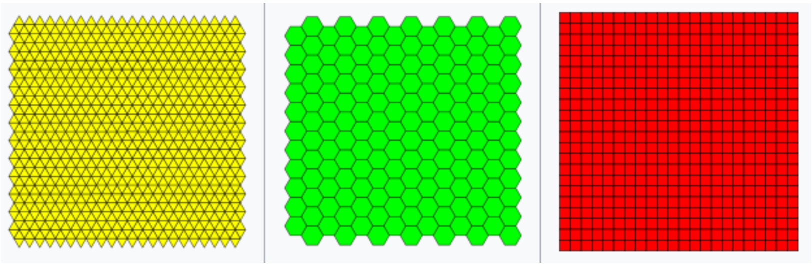
September 2021

Tessellation: Tiling the plane

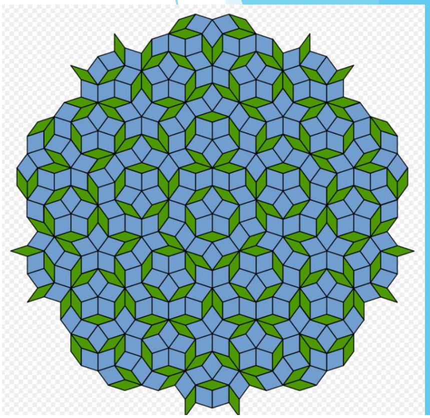
- Introduction to Tessellation (“tiling”)
  - Some examples and definitions
- Periodic tiling
  - A story of pentagonal tiling
- Aperiodic tiling
  - A story of Penrose tiling

Tessellation: Examples

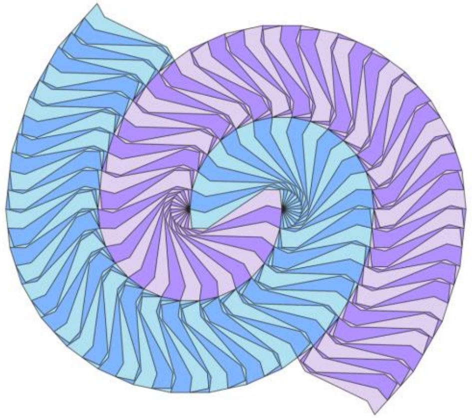
- Tessellation is the covering of a plane using one or more geometric shapes, with no overlaps and no gaps.



Regular, periodic tiling



Penrose tiling, aperiodic, using 2 shapes



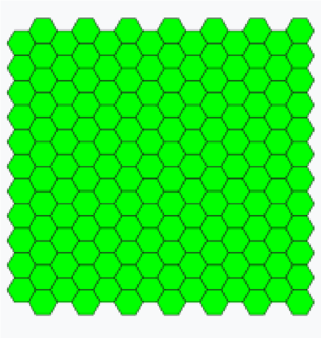
Voderberg tiling

Tiling with curved shapes

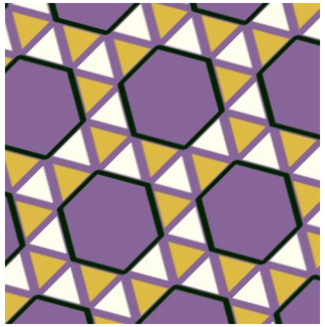
M.C. Escher, 1951

Tessellation: Some definitions

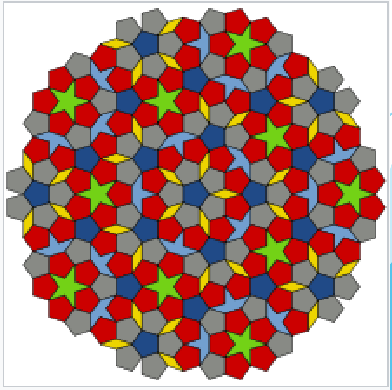
- Regular tilings: regular polygonal tiles all of the same shape.



- Semiregular tilings: regular polygonal tiles of more than one shape.



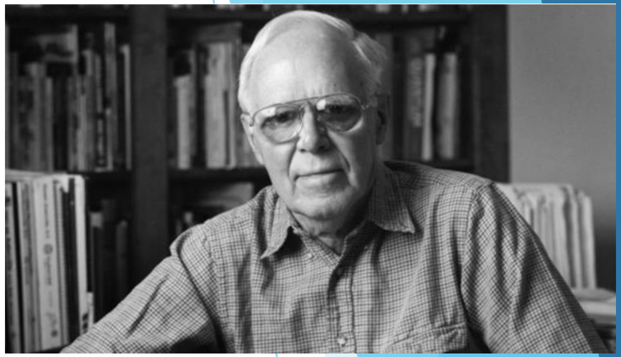
- Aperiodic tiling: uses a small set of tile shapes that cannot form a repeating pattern.



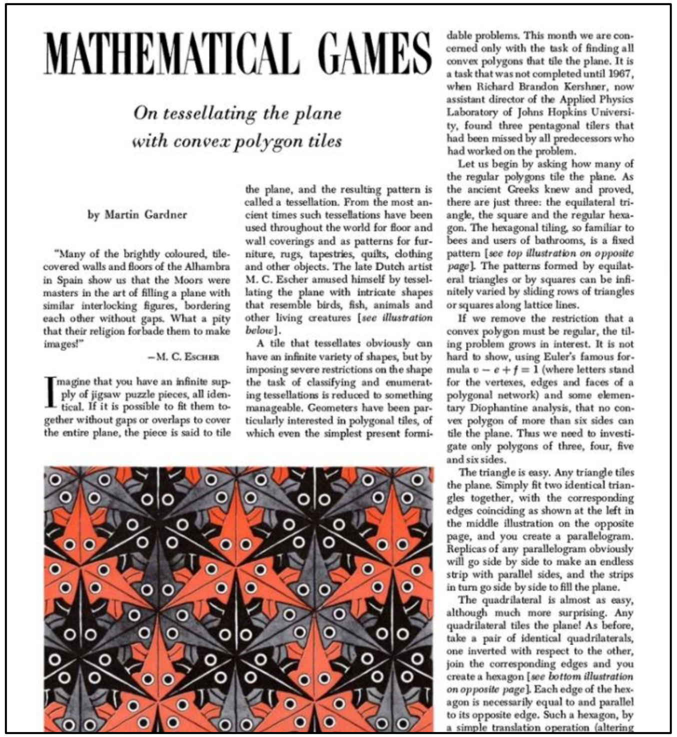
Some tiling fundamentals

- All triangles tile the plane.
- All quadrilaterals tile the plane.
  - The quadrilateral need not be convex.
- Regular pentagons cannot tile the plane.
- Some equilateral pentagons can tile the plane.
- No convex polygon with more than 6 sides can tile the plane.
- The first story focuses on convex pentagonal tiling.

A story of pentagonal tiling



- Scientific American, 1975. Martin Gardner’s column Mathematical Games
  - “On tessellating the plane with convex polygon tiles”
- Five tiling pentagons discovered by Karl Reinhardt, 1918.
- Three more discovered by R. B. Kershner, 1968.
  - Kershner claimed in this article that the set was complete with 8 pentagonal tilings.



MATHEMATICAL GAMES

On tessellating the plane with convex polygon tiles

by Martin Gardner

“Many of the brightly coloured, tile-covered walls and floors of the Alhambra in Spain show us that the Moors were masters in the art of filling a plane with similar interlocking figures, bordering each other without gaps. What a pity that their religion forbade them to make images!”

—M. C. Escher

Imagine that you have an infinite supply of jigsaw puzzle pieces, all identical. If it is possible to fit them together without gaps or overlaps to cover the entire plane, the piece is said to tile the plane, and the resulting pattern is called a tessellation. From the most ancient times such tessellations have been used throughout the world for floor and wall coverings and as patterns for furniture, rugs, tapestries, quilts, clothing and other objects. The late Dutch artist M. C. Escher amused himself by tesselating the plane with intricate shapes that resemble birds, fish, animals and other living creatures [*see illustration below*].

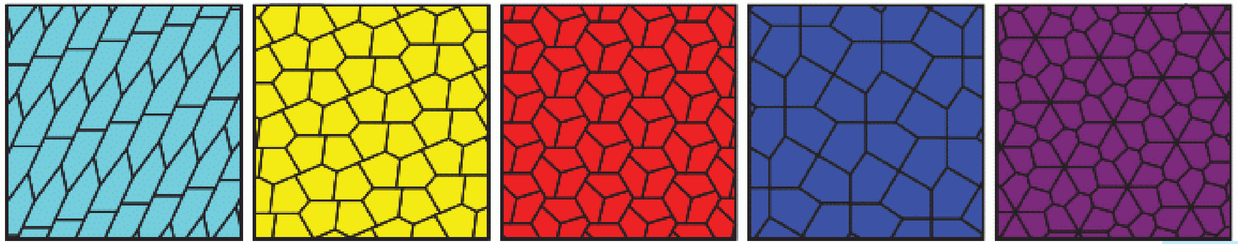
A tile that tessellates obviously can have an infinite variety of shapes, but by imposing severe restrictions on the shape the task of classifying and enumerating tessellations is reduced to something manageable. Geometers have been particularly interested in polygonal tiles, of which even the simplest present formidable problems. This month we are concerned only with the task of finding all convex polygons that tile the plane. It is a task that was not completed until 1967, when Richard Brandon Kershner, now assistant director of the Applied Physics Laboratory of Johns Hopkins University, found three pentagonal tilers that had been missed by all predecessors who had worked on the problem.

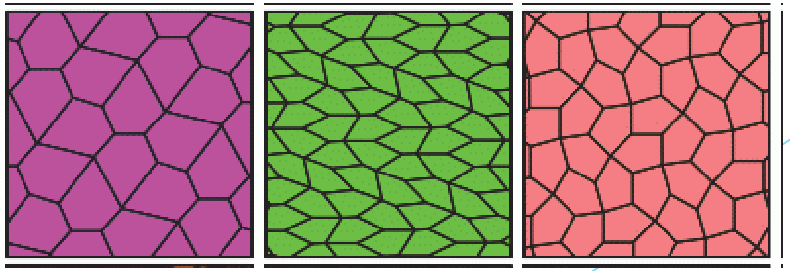
Let us begin by asking how many of the regular polygons tile the plane. As the ancient Greeks knew and proved, there are just three: the equilateral triangle, the square and the regular hexagon. The hexagonal tiling, so familiar to bees and users of bathrooms, is a fixed pattern [*see top illustration on opposite page*]. The patterns formed by equilateral triangles or by squares can be infinitely varied by sliding rows of triangles or squares along lattice lines.

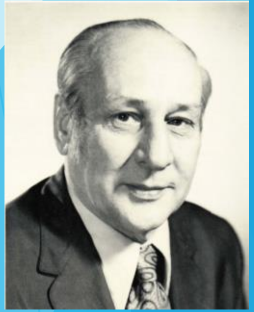
If we remove the restriction that a convex polygon must be regular, the tiling problem grows in interest. It is not hard to show, using Euler’s famous formula $v - e + f = 1$ (where letters stand for the vertexes, edges and faces of a polygonal network) and some elementary Diophantine analysis, that no convex polygon of more than six sides can tile the plane. Thus we need to investigate only polygons of three, four, five and six sides.

The triangle is easy. Any triangle tiles the plane. Simply fit two identical triangles together, with the corresponding edges coinciding as shown at the left in the middle illustration on the opposite page, and you create a parallelogram. Replicas of any parallelogram obviously will go side by side to make an endless strip with parallel sides, and the strips in turn go side by side to fill the plane.

The quadrilateral is almost as easy, although much more surprising. Any quadrilateral tiles the plane! As before, take a pair of identical quadrilaterals, one inverted with respect to the other, join the corresponding edges and you create a hexagon [*see bottom illustration on opposite page*]. Each edge of the hexagon is necessarily equal to and parallel to its opposite edge. Such a hexagon, by a simple translation operation (altering





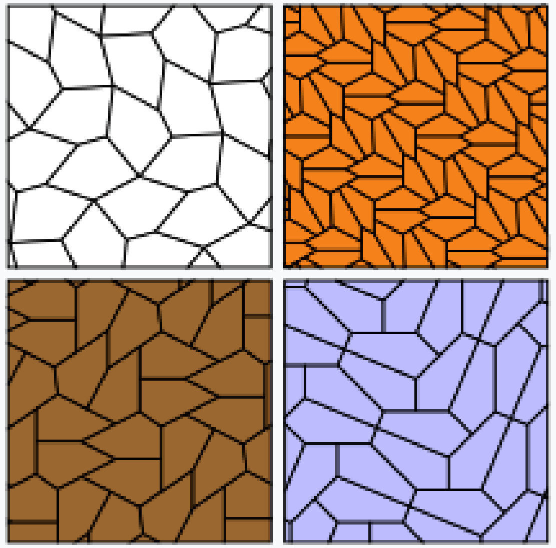


Marjorie Rice

- Marjorie Rice, a Florida housewife and mother of 6, with one year of high school math, read the tiling article in her son's copy of Scientific American.
  - She decided to investigate.
  - "by drawing diagrams on the kitchen table when no one was around and hiding them when her husband and children came home, or when friends stopped by"
  - Developed her own notation.

Marjorie Rice

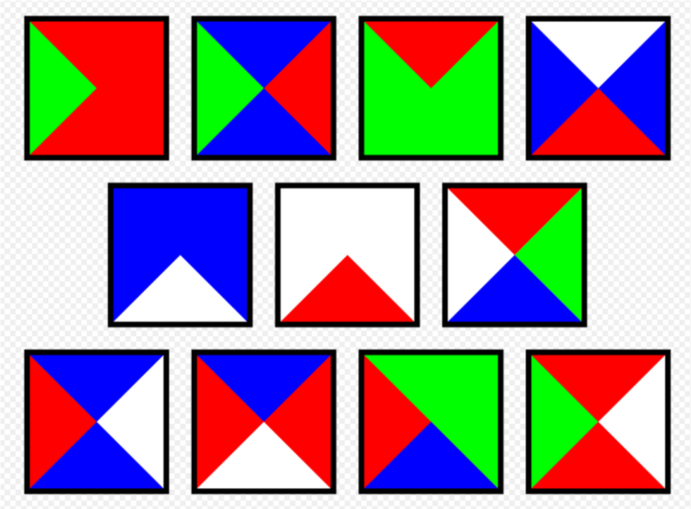
- Marjorie Rice, a Florida housewife and mother of 6, with one year of high school math, read the tiling article in her son's copy of Scientific American.
  - She decided to investigate.
  - "by drawing diagrams on the kitchen table when no one was around and hiding them when her husband and children came home, or when friends stopped by"
  - Developed her own notation.
- She discovered four more convex pentagons that tile the plane, 1976.
  - Gardner: "Fantastic achievements". He honored her in an addendum to his article.
  - Kershner immediately admitted his mistake.
- The list of convex tiling pentagons now stands at 15, with a partial proof of completeness.

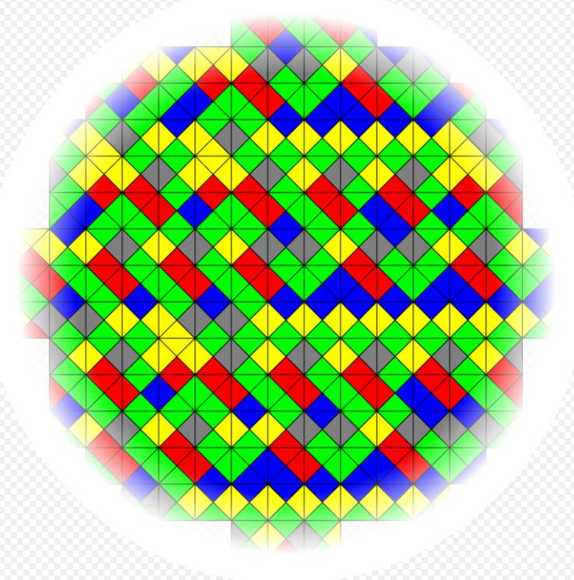


Four of Rice's pentagon tilings

Aperiodic tiling: the early days

- Does not form a repeating pattern.
- The first specific occurrence of aperiodic tilings arose in 1961, Hao Wang.
  - He first identified a set of 20,426 tiles, later reduced to 11 by Rao, 2015.

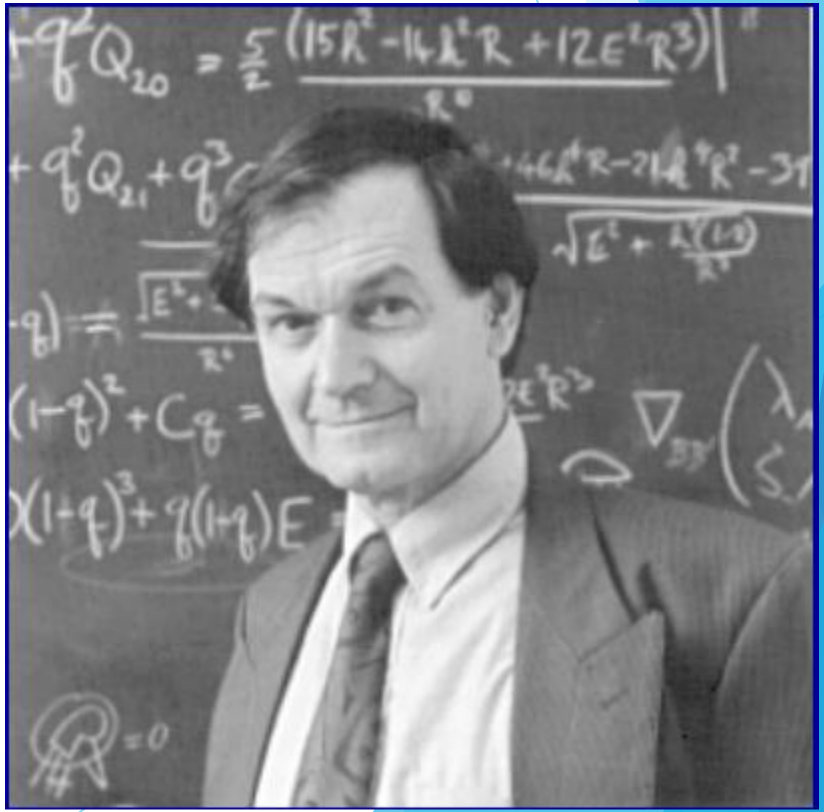




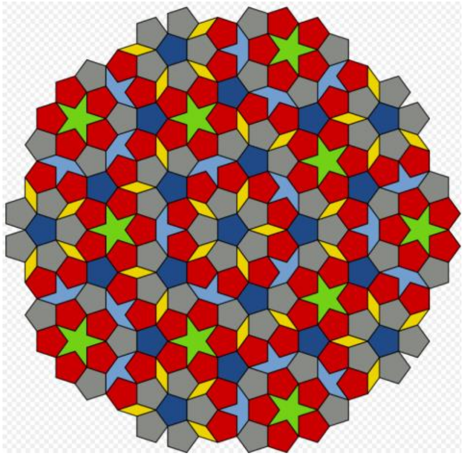
- After the discovery of quasicrystals, aperiodic tilings were studied by physicists and mathematicians.

Aperiodic tiling: Penrose

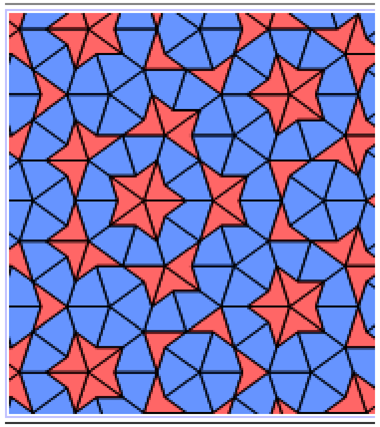
- Roger Penrose
  - Nobel Laureate in physics, 2020, with Reinhard Genzel and Andrea Ghez.
    - Black hole formation is a robust prediction of the general theory of relativity.
  - Physicist, mathematician, cosmologist, popular author. Collaborator with Stephen Hawking.
  - In 1994, Penrose was knighted for services to science.
  - Discovered several new aperiodic tilings, starting in 1974.



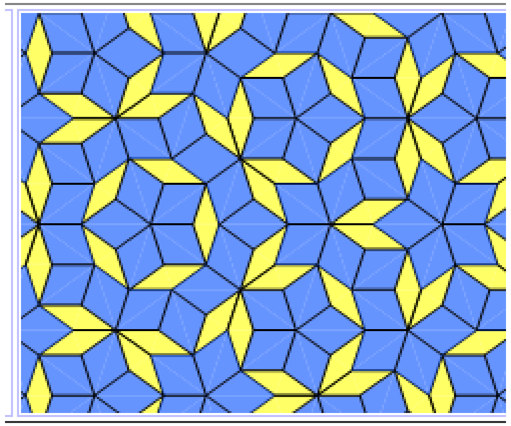
The three Penrose tilings



P1



P2 (kites and darts)

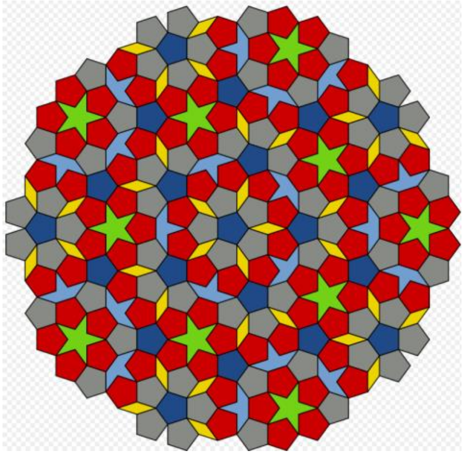


P3

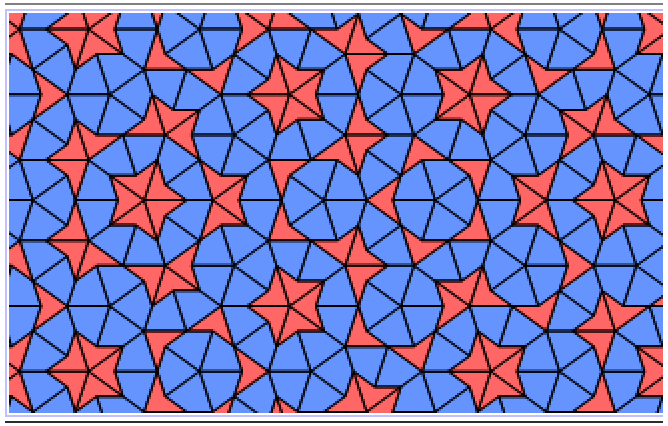
Roger Penrose in the foyer of the Mitchell Institute for Fundamental Physics and Astronomy, Texas A&M University, standing on a floor with a Penrose tiling

As few as 2 tile shapes. No edge-coloring required.

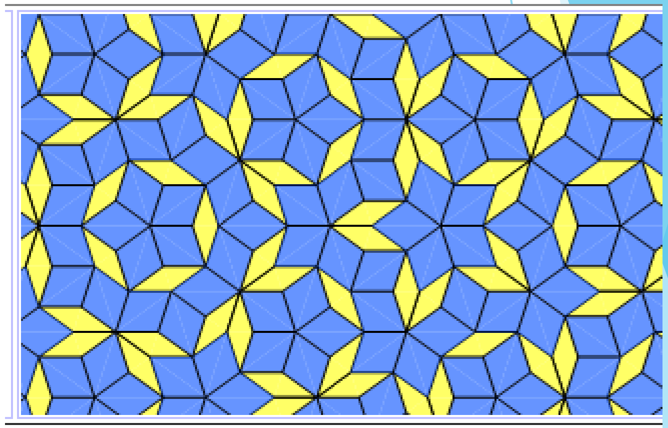
The three Penrose tilings



P1



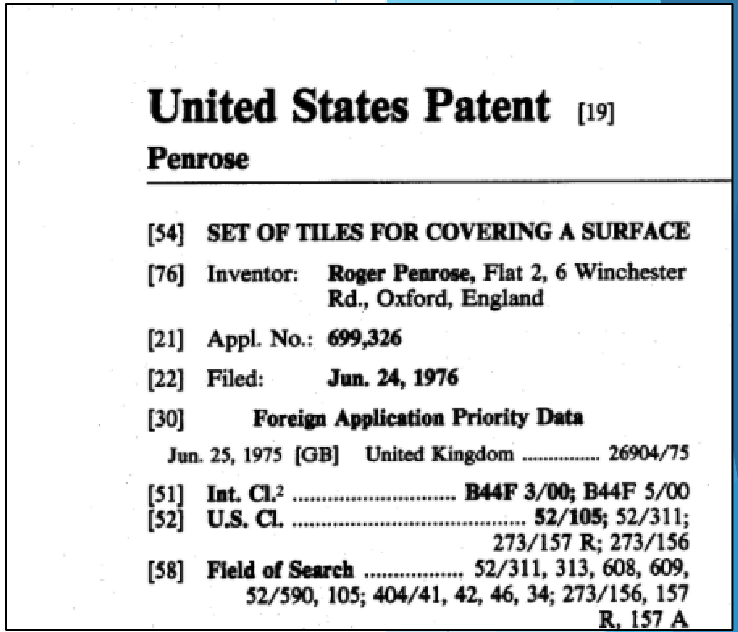
P2 (kites and darts)



P3

Yes, you can buy ceramic Penrose tiles. AND other Penrose tile products.

Penrose sues Kimberly Clark



United States Patent [19]

Penrose

[54] SET OF TILES FOR COVERING A SURFACE

[76] Inventor: **Roger Penrose**, Flat 2, 6 Winchester Rd., Oxford, England

[21] Appl. No.: **699,326**

[22] Filed: **Jun. 24, 1976**

[30] **Foreign Application Priority Data**

Jun. 25, 1975 [GB] United Kingdom 26904/75

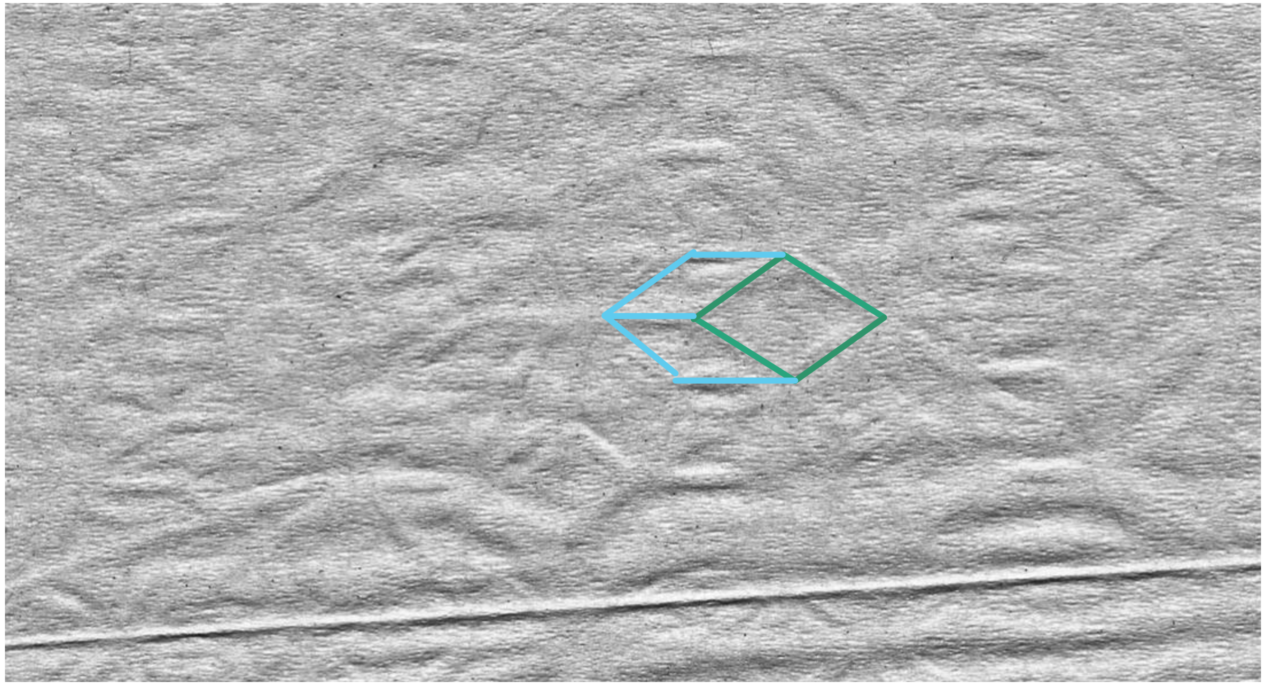
[51] **Int. Cl.**[2] **B44F 3/00; B44F 5/00**

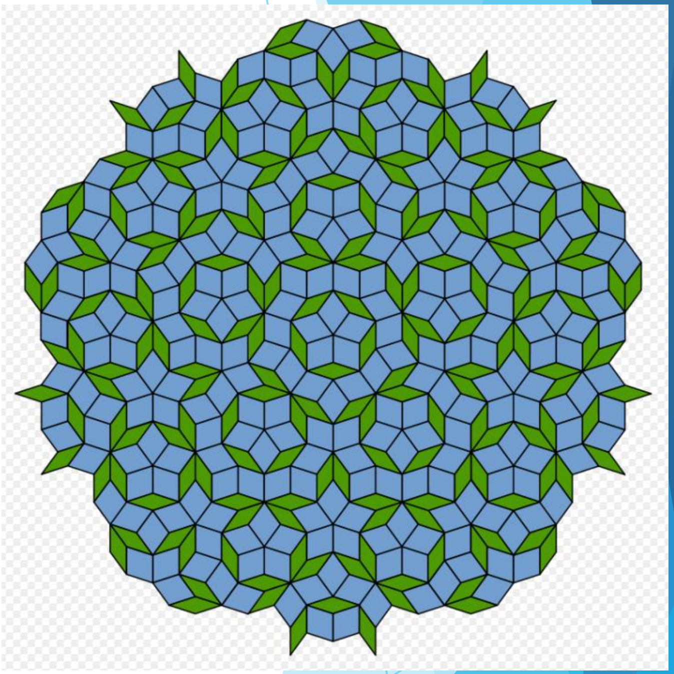
[52] **U.S. Cl.** .... **52/105**; 52/311; 273/157 R; 273/156

[58] **Field of Search** .... 52/311, 313, 608, 609, 52/590, 105; 404/41, 42, 46, 34; 273/156, 157 R, 157 A

In 1997, Penrose sued Kimberly Clark (makers of Cottonelle) for one million pounds over the use of his patented pattern used in their quilted toilet paper.

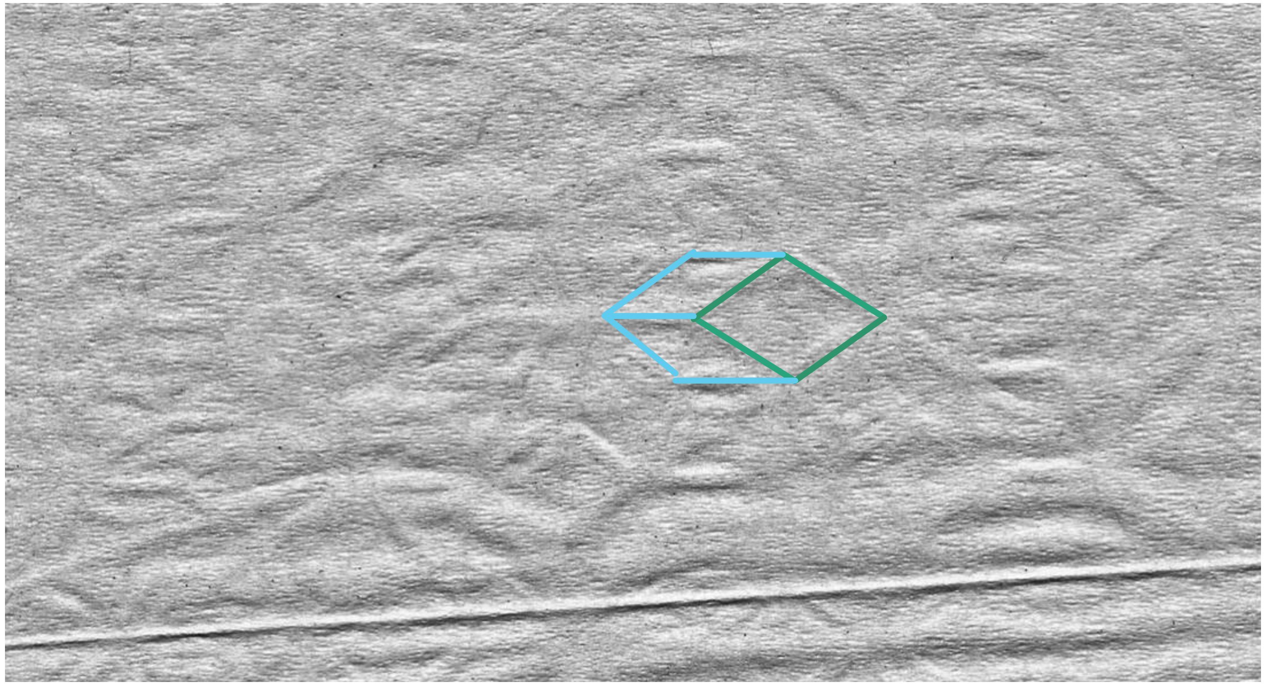
The infamous Kimberly Clark quilted toilet paper

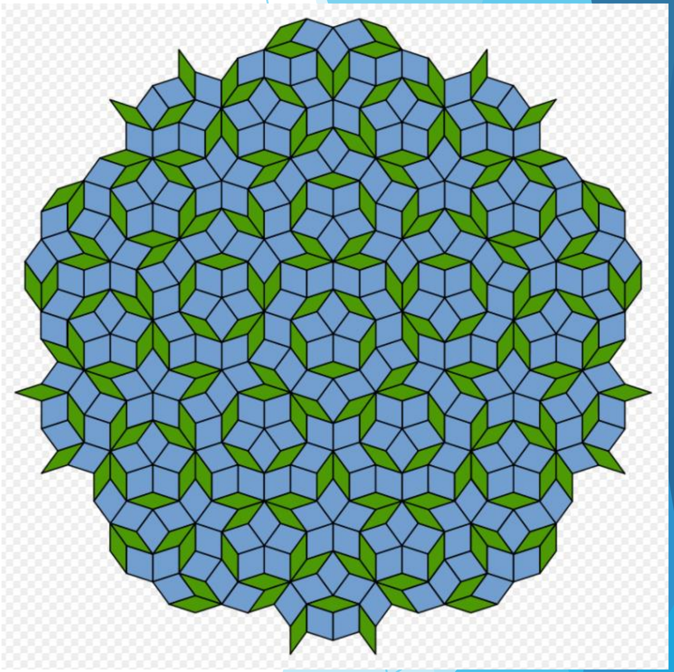




The pattern is extremely subtle.

The infamous Kimberly Clark quilted toilet paper



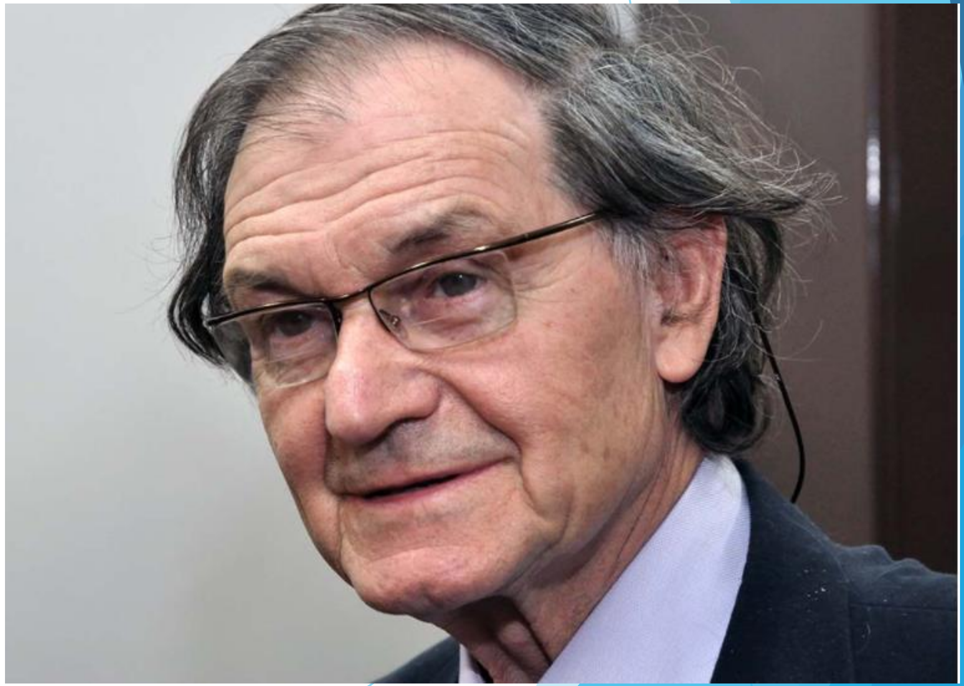


The pattern is extremely subtle.

Consider: Cottonelle by Kimberly Clark,
451 sheets, 150 feet per roll double ply. To emboss a true aperiodic quilt pattern, an embossing roller needs to be almost 48 ft in diameter.

Penrose wins

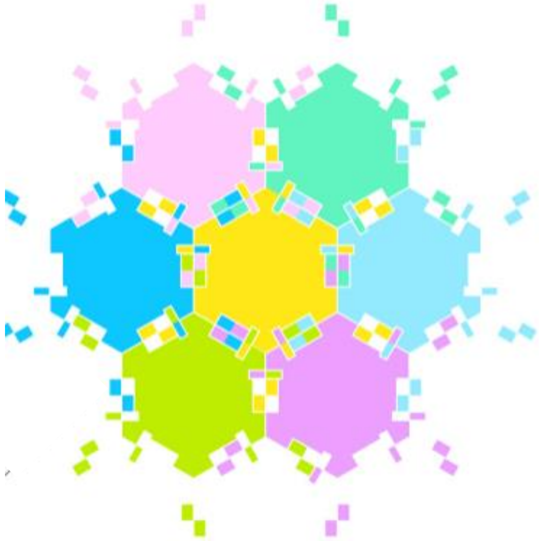
- The amount of the settlement out of court was not disclosed.
- Kimberly Clark can no longer make the Penrose toilet paper.
- Penrose: “My *cheeks* are *flushed* with joy”



Current efforts: Seeking the einstein

Does there exist one stone (einstein) that can
Tile the plane aperiodically?

Is this a tile?

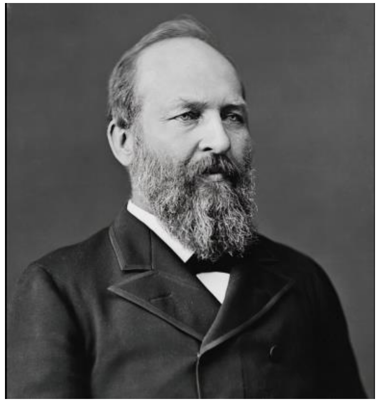


Closing thoughts

- Marjorie Rice’s name is among a long list of accomplished amateur mathematicians.
  - Wikipedia lists about 100, including:

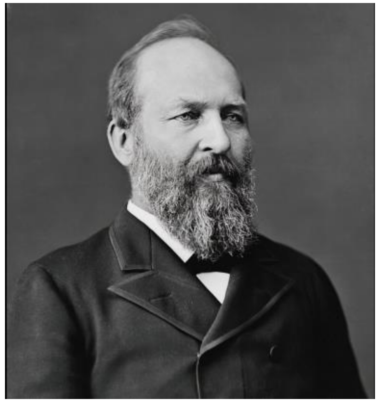
Closing thoughts

- Marjorie Rice’s name is among a long list of accomplished amateur mathematicians.
  - Wikipedia lists about 100, including:
    - James Garfield, U.S. President
      - Developed a trapezoid proof of the Pythagorean theorem



Closing thoughts

- Marjorie Rice's name is among a long list of accomplished amateur mathematicians.
  - Wikipedia lists about 100, including:



    - James Garfield, U.S. President
      - Developed a trapezoid proof of the Pythagorean theorem

    - Hedy Lamarr, actress
      - Frequency-hopping spread spectrum communication